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Theoretical Computer Science 352 (2006) 332–341
Theoretical Computer Science
www.elsevier.com/locate/tcs

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Range assignment for energy efficient broadcasting in linear radio networks

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Received 24 August 2005; received in revised form 5 November 2005; accepted 19 November 2005

Communicated by J. Diaz

Abstract

Given a set S of n radio-stations located on a d-dimensional space, a source node s ($\in S$) and an integer h ($1 \leqslant h \leqslant n-1$), the *h-hop broadcast range assignment* problem deals with assigning ranges to the members in S so that s can communicate with all other members in S in at most h-hops, and the total power consumption is minimum. The problem is known to be NP-hard for $d \geqslant 2$. We propose an $O(n^2)$ time algorithm for the one dimensional version ($d = 1$) of the problem. This is an improvement over the existing result on this problem by a factor of h [A.E.F. Clementi et al. The minimum broadcast range assignment problem on linear multi-hop wireless networks, Theoret. Comput. Sci. 299 (2003) 751–761].

Keywords: Range assignment; Broadcast; Mobile communication; Algorithm

1. Introduction

We consider the problem of assigning transmission ranges to the nodes of a linear radio-network to minimize power consumption while ensuring broadcast from a dedicated node (called source) to all other nodes in the network. A radio-network is a finite set S of *radio-stations* located on a geographical region which can communicate each other by transmitting and receiving radio signals. Each radio-station $s \in S$ is assigned a range $\rho(s)$ (a non-negative real number) for communication with other stations. A radio-station s (having $\rho(s) > 0$) can communicate (i.e., send a message) directly (i.e., in 1-*hop*) to any other station t, if the Euclidean distance between s and t is less than or equal to $\rho(s)$. If s cannot communicate directly with t due to its assigned range, then communication between them can be achieved using *multi-hop* transmission. If the maximum number of hops allowed (h) is small, then communication between a pair of radio-stations happen very quickly, but the power consumption of the entire radio-network will be high. On the other hand, if h is large then the power consumption decreases, but communication delay takes place. The tradeoff between the power consumption of the radio-network and the maximum number of hops needed between a communicating pair of radio-stations are studied extensively in [6,7]. As in [5], we assume that $power(s) = (\rho(s))^2$. Thus the total power requirement (cost) of a range assignment $\mathcal{R} = \{\rho(s) \mid s \in S\}$ is $cost(\mathcal{R}) = \sum_{s\in S} power(s) = \sum_{s \in S} (\rho(s))^2$.

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0304-3975/$ - see front matter
doi:10.1016/j.tcs.2005.11.046

The objective of *h-hop broadcast range assignment* problem is to assign transmission ranges $\rho(t)$ to the radio-stations $t \in S$ so that a dedicated radio-station (say $s \in S$) can transmit message to all other radio-stations using at most *h-hops*, and the total power consumption of the entire network is minimum. For $h \geqslant 2$, the problem is NP-hard even in 2D [2,3]. For the 1D version of the problem, a dynamic programming based algorithm is proposed in [5]. It runs in $O(hn^2)$ time, where $n = |S|$. We improve the time complexity result of the problem proposed in [5]. Our algorithm is simple, and it runs in $O(n^2)$ time and $O(hn)$ space.

In spite of the fact that the model considered in this paper is simple, it is very much useful in studying road traffic information system where the vehicles follow roads and messages are to be broadcasted along lanes. Typically, the curvature of the road is small in comparison to the transmission range so that we can consider that the vehicles are moving on a line [4]. Linear radio networks have been observed to be important in several recent studies [4–7].

2. Structure of optimal broadcast range assignment

We assume that the radio-stations $S = \{s_1, s_2, \ldots, s_n\}$ are ordered on the x-axis from left to right, with s_1 positioned at 0 (the origin). The position of s_i will be denoted by $x(s_i)$. Thus, the distance between two radio-stations s_i and s_j is $\delta(s_i, s_j) = |x(s_i) - x(s_j)|$. We will use $\mathcal{C}(S, s, h)$ to denote the minimum among the costs of the range assignments of the members in S for broadcasting message from the source station s ($\in S$) to all other radio-stations in S using at most h-hops. There may be several range assignments of S having cost $\mathcal{C}(S, s, h)$. We will use $\mathcal{R}(S, s, h)$ to denote one such range assignment, and will refer it as *optimal range assignment*.

Definition 1. In a h-hop broadcast range assignment, a right-bridge $\overleftarrow{s_\ell s_r}$ corresponds to a pair of radio-stations (s_ℓ, s_r) such that s_ℓ is to the left of s, s_r is to the right of s, and $\delta(s_\ell, s_r) \leqslant \rho(s_r) < \delta(s_{\ell-1}, s_r)$. In other words, s_r can communicate with s_ℓ in 1-hop due to its assigned range, but it cannot communicate with $s_{\ell-1}$ in 1-hop.

Definition 2. In a h-hop broadcast range assignment, a right-bridge $\overleftarrow{s_\ell s_r}$ (if exists) is called functional, if there exists a radio-station $s_i \in S$ such that the minimum number of hops among all the paths from s to s_i that avoids the 1-hop communication $\overleftarrow{s_\ell s_r}$, is greater than h.

Similarly, one can define a *left-bridge* $\overrightarrow{s_\ell s_r}$ and a *functional left-bridge* in a h-hops range assignment, where s_ℓ and s_r are respectively to the left and right of s.

Theorem 1 (*Clementi et al. [5]*). *Given a set of radio-stations $S = \{s_1, s_2, \ldots, s_n\}$, a source node $s \in S$, and an integer h $(1 \leqslant h \leqslant n-1)$, the optimal h-hop broadcast range assignment $\mathcal{R}(S, s, h)$ contains at most one functional bridge.*

The algorithm proposed in [5] solves the problem in three phases. It computes optimal solutions having (i) no functional (left/right) bridge, (ii) one functional left-bridge only, and (iii) one functional right-bridge only. Finally, the one having minimum total cost is reported. Our algorithm is based on the same principle as in [5], but it considers the geometry of the range assignment for obtaining the optimal solution in each of the three cases mentioned in (i)–(iii) in a careful manner, and this leads to an algorithm with improved time complexity.

3. Geometric properties

Lemma 1. *In a linear ordered set of radio-stations $\{s_a, s_{a+1}, \ldots, s_b\} \subseteq S$, if the source station is at one end of the above set (say s_a), then for any $1 \leqslant \mu \leqslant b-a$, an optimum μ-hop broadcast range assignment $\mathcal{R}(\{s_a, s_{a+1}, \ldots, s_b\}, s_a, \mu)$ should satisfy $\sum_{k=a}^{b-1} \rho(s_k) = x(s_b) - x(s_a)$.*

Proof. Consider the μ-hop path for communication from s_a to s_b as shown in Fig. 1(a). Note that, one can reduce the total cost of range assignment $(\sum_{k=a}^{b-1} (\rho(s_k))^2)$ by setting $\rho(s_{j+1}) = \rho(s_{j+2}) = \ldots = \rho(s_i) = 0$ (see Fig. 1(b)). This maintains μ-hop connections from s_a to all other nodes in the set. $\square$

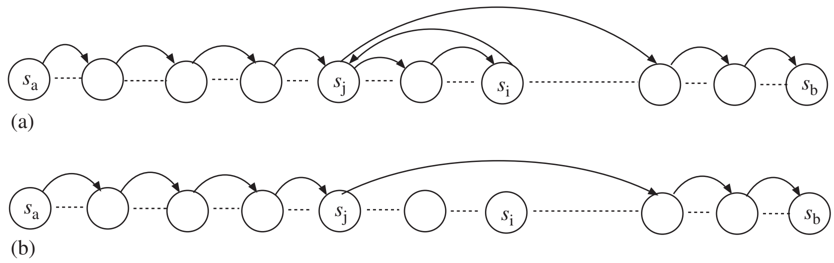

Fig. 1. Proof of Lemma 1.

Lemma 2. *For a set of radio-stations* $S = \{s_1, s_2, \ldots, s_n\}$, $\mathcal{C}(S, s_1, \mu) = \mathcal{C}(S, s_n, \mu)$.

Proof. Let $\{a_0, a_1, \ldots, a_{\mu-1}\} \subseteq S$ be the sequence of radio-stations having non-zero ranges in $\mathcal{R}(S, s_1, \mu)$. Here $a_0 = s_1$, and let us denote $a_\mu = s_n$. By Lemma 1, $\rho(a_i) = x(a_{i+1}) - x(a_i)$, for $i = 0, 1, \ldots, \mu - 1$. A feasible range assignment for communicating from s_n to s_1 using μ-hops is $\rho(a_i) = x(a_i) - x(a_{i-1})$, for $i = 1, 2, \ldots, \mu$, and its cost is same as $\mathcal{C}(S, s_1, \mu)$. Thus $\mathcal{C}(S, s_n, \mu) \leqslant \mathcal{C}(S, s_1, \mu)$. Following the same method, it can be shown that $\mathcal{C}(S, s_1, \mu) \leqslant \mathcal{C}(S, s_n, \mu)$. Hence the result follows. $\square$

Lemma 3. *In an optimum* μ*-hop broadcast range assignment* $\mathcal{R}(S, s_1, \mu)$, *if the range assigned to* s_1 *is* $\rho(s_1) = \delta(s_1, s_j)$ *for some* $j > 1$, *then there exists a* $\mathcal{R}(S \backslash \{s_1\}, s_2, \mu)$, *where* $\rho(s_2) \geqslant \delta(s_2, s_j)$.

Proof. In $\mathcal{R}(S, s_1, \mu)$, $\rho(s_1) = \delta(s_1, s_j)$ implies that $\rho(s_2) = \rho(s_3) = \cdots = \rho(s_{j-1}) = 0$. Thus, if $\mathcal{C}(S, s_1, \mu) = c$ then $\mathcal{C}(S \setminus \{s_1, s_2, \ldots, s_{j-1}\}, s_j, \mu - 1) = c - (\delta(s_1, s_j))^2$. In other words, the range assignments of the radio-stations $S \backslash \{s_1, s_2, \ldots, s_{j-1}\}$ in $\mathcal{R}(S, s_1, \mu)$ are such that, it supports broadcasting from s_j to all the radio-stations $\{s_{j+1}, \ldots, s_n\}$ in $(\mu - 1)$-hops with minimum cost.

Now, let us assume that the range assigned to s_2 in $\mathcal{R}(S \setminus \{s_1\}, s_2, \mu)$ is $\rho(s_2) = \delta(s_2, s_k)$. We need to prove that $k \geqslant j$.

Let us assume that $\mathcal{C}(S \setminus \{s_1\}, s_2, \mu) = c'$. This implies, $\mathcal{C}(S \setminus \{s_1, s_2, \ldots, s_{k-1}\}, s_k, \mu - 1) = c' - (\delta(s_2, s_k))^2$. Thus, $\{\delta(s_1, s_k), \underbrace{0, 0, \ldots, 0}_{k-2}, \mathcal{R}(S \setminus \{s_1, s_2, \ldots, s_{k-1}\}, s_k, \mu - 1)\}$ is a feasible range assignment (may not be optimum) for the μ-hop broadcast from s_1 to all the nodes in $S \setminus \{s_1\}$, and its cost is equal to $(\delta(s_1, s_k))^2 + (c' - (\delta(s_2, s_k))^2) \geqslant c$. This implies, $c - c' \leqslant (\delta(s_1, s_2))^2 + 2\delta(s_1, s_2)\delta(s_2, s_k)$.

By a similar argument, $\{\delta(s_2, s_j), \underbrace{0, 0, \ldots, 0}_{j-3}, \mathcal{R}(S \setminus \{s_1, s_2, \ldots, s_{j-1}\}, s_j, \mu - 1)\}$ is a feasible range assignment for the μ-hop broadcast from s_2 to all the nodes in $S \setminus \{s_1, s_2\}$, and its cost is equal to $(\delta(s_2, s_j))^2 + (c - (\delta(s_1, s_j))^2) \geqslant c'$. This implies, $c - c' \geqslant (\delta(s_1, s_2))^2 + 2\delta(s_1, s_2)\delta(s_2, s_j)$.

Combining these two inequalities, we have

$$(\delta(s_1, s_2))^2 + 2\delta(s_1, s_2)\delta(s_2, s_j) \leqslant c - c' \leqslant (\delta(s_1, s_2))^2 + 2\delta(s_1, s_2)\delta(s_2, s_k).$$

This implies $k \geqslant j$. $\square$

In the following lemma, we prove that if we increase the number of allowable hops for broadcasting from a fixed radio-station, say s_1, to all the radio-stations to its right, then the gain in the cost obtained in two consecutive steps are monotonically decreasing.

Lemma 4. $\mathcal{C}(S, s_1, \mu) - \mathcal{C}(S, s_1, \mu + 1) \geqslant \mathcal{C}(S, s_1, \mu + 1) - \mathcal{C}(S, s_1, \mu + 2)$.

Proof. Let $A = \{a_0 = s_1, a_1, a_2, \ldots, a_{\mu-1}\}$ denote the subsequence (radio stations) of S having non-zero ranges in $\mathcal{R}(S, s_1, \mu)$. We use a_μ to denote the radio-station s_n and $cost(A)$ to denote $\mathcal{C}(S, s_1, \mu)$. Here, the range assigned to $a_i \in A$ is $(x(a_{i+1}) - x(a_i))$ for $i = 0, 1, 2, \ldots, \mu - 1$. Again, let $B = \{b_0 = s_1, b_1, b_2, \ldots, b_{\mu+1}\}$ denotes the set of

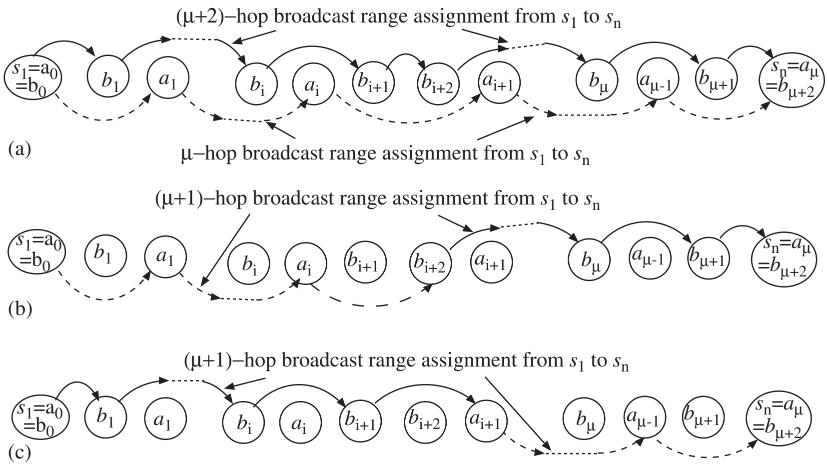

Fig. 2. Proof of Lemma 4.

radio stations having non-zero ranges in $\mathcal{R}(S, s_1, \mu+2)$, i.e., $cost(B) = \mathcal{C}(S, s_1, \mu+2)$. As earlier, s_n is denoted by $b_{\mu+2}$, and the ranges assigned to $b_i (\in B)$ are $(x(b_{i+1}) - x(b_i))$ for $i = 0, 1, 2, \ldots, \mu+1$. The two range assignments (A and B) are shown in Fig. 2(a) using solid and dashed lines.

Observe that, $x(a_0) - x(b_1) < 0$, and $x(a_\mu) - x(b_{\mu+1}) > 0$. This implies, there exists at least one $i \in \{1, 2, \ldots, \mu-1\}$ such that $x(a_i) - x(b_{i+1}) \leqslant 0$ and $x(a_{i+1}) - x(b_{i+2}) \geqslant 0$. We consider the smallest $i \geqslant 0$ such that $x(a_{i+1}) - x(b_{i+2}) \geqslant 0$, and construct two subsequences of radio stations $C = \{a_0 = b_0 = s_1, a_1, \ldots, a_i, b_{i+2}, b_{i+3}, \ldots, b_{\mu+1}\}$ and $D = \{a_0 = b_0 = s_1, b_1, b_2, \ldots, b_{i+1}, a_{i+1}, a_{i+2}, \ldots, a_{\mu-1}\}$, each of length $\mu+1$. The ranges assigned to the members in C and D are, respectively,

- $\{x(a_1) - x(a_0), \ldots, x(a_i) - x(a_{i-1}), x(b_{i+2}) - x(a_i), x(b_{i+3}) - x(b_{i+2}), \ldots, x(b_{\mu+2}) - x(b_{\mu+1})\}$ (see Fig. 2(b)), and
- $\{x(b_1) - x(b_0), \ldots, x(b_{i+1}) - x(b_i), x(a_{i+1}) - x(b_{i+1}), x(a_{i+2}) - x(a_{i+1}), \ldots, x(a_\mu) - x(a_{\mu-1})\}$ (see Fig. 2(c)).

The corresponding costs of the range assignments are

$$cost(C) = \sum_{j=0}^{j=i-1} (x(a_{j+1}) - x(a_j))^2 + (x(b_{i+2}) - x(a_i))^2 + \sum_{j=i+2}^{j=\mu+1} (x(b_{j+1}) - x(b_j))^2$$

and

$$cost(D) = \sum_{j=0}^{j=i} (x(b_{j+1}) - x(b_j))^2 + (x(a_{i+1}) - x(b_{i+1}))^2 + \sum_{j=i+1}^{j=\mu-1} (x(a_{j+1}) - x(a_j))^2.$$

Thus,

$$\begin{aligned} cost(C) + cost(D) = &\left(\sum_{j=0}^{j=\mu-1} (x(a_{j+1}) - x(a_j))^2 - (x(a_{i+1}) - x(a_i))^2 \right) \\ &+ \left(\sum_{j=0}^{j=\mu+1} (x(b_{j+1}) - x(b_j))^2 - (x(b_{i+2}) - x(b_{i+1}))^2 \right) \\ &+ (x(b_{i+2}) - x(a_i))^2 + (x(a_{i+1}) - x(b_{i+1}))^2 \\ = &\, cost(A) + cost(B) + 2(x(a_i) - x(b_{i+1}))(x(a_{i+1}) - x(b_{i+2})) \leqslant cost(A) + cost(B) \end{aligned}$$

(due to the choice of i as mentioned above).

Let O indicate a subsequence of $\mu+1$ radio-stations with non-zero range assignments such that s_1 can send message to s_n in $\mu+1$ hops (or equivalently to all members in S in at most $\mu+1$ hops) and the cost of range assignment is minimum, i.e., $cost(O) = \mathcal{C}(S, s_1, \mu+1)$. Thus we have, $2 \times cost(O) \leqslant cost(C) + cost(D) \leqslant cost(A) + cost(B)$. □

4. Algorithms

Let $s_\alpha \in S$ be the given source station (not necessarily the left-most/right-most in the ordering of S). Our algorithm for broadcasting from s_α to all other radio-stations $s_j \in S$ consists of three phases. Phase 1 prepares four initial matrices. These are used in Phases 2 and 3 for computing optimal solution with no functional bridge, and exactly one functional bridge, respectively.

For notational convenience, if the source radio-station (say s_a) is at one end of a linearly ordered destination stations $\{s_a, s_{a+1}, \ldots, s_b\}$, then we will use $R(s_b, s_a, \mu)$ and $\mathcal{C}(s_b, s_a, \mu)$ to denote the optimal range assignment $\mathcal{R}(\{s_a, s_{a+1}, \ldots, s_b\}, s_a, \mu)$ and $\mathcal{C}(\{s_a, s_{a+1}, \ldots, s_b\}, s_a, \mu)$, respectively.

4.1. Phase 1

In this phase, we prepare the following four initial matrices. These will be extensively used in Phases 2 and 3. Recall that s_α is the source station.

M_1: It is a $h \times (\alpha - 1)$ matrix. Its (m, j)th element $(1 \leqslant j < \alpha)$ indicates the optimum cost of sending message from s_j to s_α (source station) using m hops. In other words, $M_1[m, j] = \mathcal{C}(s_\alpha, s_j, m)$, where $1 \leqslant m \leqslant h$ and $1 \leqslant j < \alpha$.

M_2: It is a $h \times (\alpha - 1)$ matrix. Its (m, j)th element $(1 < j \leqslant \alpha)$ indicates the optimum cost of sending message from s_j to s_1 (left-most radio-station in S) using m hops. In other words, $M_2[m, j] = \mathcal{C}(s_1, s_j, m)$, where $1 \leqslant m \leqslant h$ and $1 < j \leqslant \alpha$.

M_3: It is a $h \times (n - \alpha)$ matrix. Its (m, j)th element $(\alpha < j \leqslant n)$ indicates the optimum cost of sending message from s_j to s_α using m hops. In other words, $M_3[m, j] = \mathcal{C}(s_\alpha, s_j, m)$, where $1 \leqslant m \leqslant h$ and $\alpha < j \leqslant n$.

M_4: It is a $h \times (n - \alpha)$ matrix. Its (m, j)th element $(\alpha \leqslant j < n)$ indicates the optimum cost of sending message from s_j to s_n (right-most radio-station) using m hops. In other words, $M_4[m, j] = \mathcal{C}(s_n, s_j, m)$, where $1 \leqslant m \leqslant h$ and $\alpha \leqslant j < n$.

Note that, the columns of M_1 are indexed as $[1, 2, \ldots, \alpha-1]$, whereas those in M_2 are indexed as $[2, 3, \ldots, \alpha]$. Similarly, the columns of M_3 are indexed as $[\alpha + 1, \alpha + 2, \ldots, n]$, whereas those in M_4 are indexed as $[\alpha, \alpha + 1, \ldots, n - 1]$. We explain an incremental approach (in terms of hops) for constructing M_1. Similar procedure works for constructing the other three matrices.

Each entry of the matrix M_1 contains a tuple (χ, ptr), where the χ field of $M_1[m, j]$ contains $\mathcal{C}(s_\alpha, s_j, m)$, and its ptr field is an integer which contains the index of the first radio-station (after s_j) on the m-hop path from s_j to s_α. We will interchangeably use, $M_1[m, j]$ and $M_1[m, j].\chi$ to denote $\mathcal{C}(s_\alpha, s_j, m)$. After computing up to row m of the matrix M_1, the elements in the $(m + 1)$th row can easily be obtained as follows:

Consider an intermediate matrix A of size $(\alpha - 1) \times (\alpha - 1)$. Its (j, k)th element contains the cost of $(m + 1)$-hop communication from s_j to s_α with first hop at s_k. Thus, $A[j, k] = (\delta(s_j, s_k))^2 + M_1[m, k]$. After computing the matrix A, we compute $M_1[m + 1, j].\chi = Min_{k=j+1}^{\alpha-1} A[j, k]$, and $M_1[m + 1, j].ptr$ will contain the value of k for which $A[j, k]$ is contributed to $M_1[m + 1, j].\chi$.

Straightforward application of the above method needs $O(\alpha^2)$ time. But, Lemma 3 says that, if in the optimum $(m+1)$-hop path from s_j to s_α with first hops at node s_k, then for any node $s_{j'}$ with $j' > j$, the optimum $(m + 1)$-hop path from s'_j to s_α with first hops at node $s_{k'}$ and $k' \geqslant k$. A simple method for computing the minimum of every row in the matrix A (without enumerating all the entries in A) needs a total of $O(\alpha \log \alpha)$ time as follows:

Compute all the entries in the $\frac{\alpha}{2}$th row of the matrix A, and find the minimum. Let it corresponds to $A[\frac{\alpha}{2}, \beta]$. Next, compute the minimum entry in $\frac{\alpha}{4}$th row of A by considering $\{A[\frac{\alpha}{4}, j], j = 1, 2, \ldots, \beta\}$, and compute the minimum entry in $\frac{3\alpha}{4}$th row of A by considering $\{A[\frac{3\alpha}{4}, j], j = \beta, \beta + 1, \ldots, \alpha - 1\}$. The process continues until all the rows of A are considered.

Definition 3 (Aggarwal and Klawe [1]). A matrix M is said to be *monotone* if for every j, k, j', k' with $j < j', k < k'$, if $M[j, k] \geqslant M[j, k']$ then $M[j', k] \geqslant M[j', k']$.

Lemma 5. *The matrix A is a monotone matrix.*

Proof. Given $A[j,k] \geqslant A[j,k']$, where $A[j,k] = (\delta(s_j, s_k))^2 + M_1[m,k]$ and $A[j,k'] = (\delta(s_j, s_{k'}))^2 + M_1[m,k']$. Thus, $M_1[m,k] - M_1[m,k'] \geqslant (\delta(s_j, s_{k'}))^2 - (\delta(s_j, s_k))^2$.

Now,

$$\begin{aligned} A[j',k] - A[j',k'] &= (\delta(s_{j'}, s_k))^2 - (\delta(s_{j'}, s_{k'}))^2 + M_1[m,k] - M_1[m,k'] \\ &\geqslant (\delta(s_{j'}, s_k))^2 - (\delta(s_{j'}, s_{k'}))^2 + (\delta(s_j, s_{k'}))^2 - (\delta(s_j, s_k))^2 \geqslant 0 \end{aligned}$$

(on simplification). □

A recursive algorithm for monotone matrix searching is described in [1], which can compute the minimum entry in each row of a $\alpha \times \alpha$ monotone matrix in $O(\alpha)$ time provided each entry of the matrix can be computed in $O(1)$ time. Using that algorithm, the matrix M_1 can be computed in $O(\alpha \times h)$ time.

Lemma 6. *Phase* 1 *needs* $O(nh)$ *time.*

Proof. Follows from the fact that M_1, M_2 can be constructed in $O(\alpha \times h)$ time, and M_3, M_4 needs $O((n-\alpha) \times h)$ time. □

4.2. Phase 2

In this phase, we compute the optimal functional bridge-free solution for broadcasting message from s_α to the other nodes in S. Here, the range to be assigned to s_α is at least $Max(\delta(s_\alpha, s_{\alpha-1}), \delta(s_\alpha, s_{\alpha+1}))$.

Without loss of generality, we assume that $\delta(s_\alpha, s_{\alpha-1}) \leqslant \delta(s_\alpha, s_{\alpha+1})$. Thus, $\rho(s_\alpha)$ is initially assigned to $\delta(s_\alpha, s_{\alpha+1})$, and let s_k $(k < \alpha)$ be the farthest radio-station such that s_α can communicate with s_k in 1-hop (i.e., $\delta(s_k, s_\alpha) \leqslant \delta(s_\alpha, s_{\alpha+1})) < \delta(s_{k-1}, s_\alpha)$). If we use $\mathcal{R}(S, s_\alpha, h | \rho(s_\alpha) = d)$ to denote the optimum range assignment for the h-hop broadcasting from s_α to all the nodes in S subject to the condition that the range assigned to s_α is d, then

$$\begin{aligned} &\mathcal{R}(S, s_\alpha, h | \rho(s_\alpha) = \delta(s_\alpha, s_{\alpha+1})) \\ &\quad = \{\mathcal{R}(\{s_1, \ldots, s_k\}, s_k, h-1), \underbrace{0, 0, \ldots, 0}_{\alpha-k-1}, \delta(s_\alpha, s_{\alpha+1}), \mathcal{R}(S \setminus \{s_1, \ldots, s_\alpha\}, s_{\alpha+1}, h-1)\}, \\ &\quad = \{R(s_1, s_k, h-1), \underbrace{0, 0, \ldots, 0}_{\alpha-k-1}, \delta(s_\alpha, s_{\alpha+1}), R(s_n, s_{\alpha+1}, h-1)\} \end{aligned}$$

and its cost is

$$\mathcal{C}^* = \mathcal{C}(S, s_\alpha, h | \rho(s_\alpha) = \delta(s_\alpha, s_{\alpha+1})) = (\delta(s_\alpha, s_{\alpha+1}))^2 + M_2[h-1, k] + M_4[h-1, \alpha+1].$$

This can be computed in $O(1)$ time using the matrices M_2 and M_4. We use two temporary variables TEMP_Cost and TEMP_id to store $\mathcal{C}^*$ and $s_{\alpha+1}$.

Next, we increment $\rho(s_\alpha)$ to $Min(\delta(s_\alpha, s_{k-1}), \delta(s_\alpha, s_{\alpha+2}))$, and apply the same procedure to calculate the optimum cost of the h-hop broadcast from s_α. This may cause update of TEMP_Cost and TEMP_id. The same procedure is repeated by incrementing $\rho(s_\alpha)$ to its next choice in the set $\{\delta(s_\alpha, s_k), k = 1, 2, \ldots, \alpha-1\} \cup \{\delta(s_\alpha, s_j), j = k+1, \ldots, n\}$ so that it can communicate directly with one more node than its previous choice. At each step, the TEMP_Cost and TEMP_id are adequately updated. Thus, the procedure is repeated for $O(n)$ times, and the time complexity of this phase is $O(n)$.

4.3. Phase 3

In this phase, we compute an optimal range assignment for the h-hop broadcasting from s_α to all other nodes in S where the solution contains a functional *right-bridge*. Similar method will be adopted to compute the optimal solution with one functional *left-bridge*. The one having minimum cost is chosen as the optimal solution obtained in this phase.

Let us consider a range assignment which includes a right-bridge $\overleftarrow{s_i s_j}$, $i < \alpha < j$. Let s_i be such that $\delta(s_j, s_k) \leqslant \delta(s_j, s_i) < \delta(s_j, s_{k+1})$, $k \geqslant j$. This can be realized in the following two ways:

Scheme 1. Assign $\rho(s_j) = \delta(s_j, s_i)$.

Scheme 2. If $\delta(s_j, s_k) < \delta(s_j, s_i) < \delta(s_j, s_{k+1}) < \delta(s_j, s_{i-1})$, then assign $\rho(s_j) = \delta(s_j, s_{k+1})$.

We assume that s_j is reached from s_α using m hops. Thus, using Scheme 1, h-hops connection from s_α to all the nodes in S is achieved by (i) reaching s_1 from s_i in $(h-m-1)$ hops, and (ii) reaching s_n from s_k in $(h-m-1)$ hops. Here the cost of range assignment is $B_1 = \mathcal{C}(s_j, s_\alpha, m) + (\delta(s_i, s_j))^2 + \mathcal{C}(s_1, s_i, h-m-1) + \mathcal{C}(s_n, s_k, h-m-1)$.

In Scheme 2, s_j can directly communicate to s_{k+1} to the right, and s_i to the left. Thus, the h-hop connection from s_α to all the nodes in S is established by (i) reaching s_1 from s_i in $(h-m-1)$ hops, and (ii) reaching s_n from s_{k+1} in $(h-m-1)$ hops. Here the cost of range assignment is $B_2 = \mathcal{C}(s_j, s_\alpha, m) + (\delta(s_j, s_{k+1}))^2 + \mathcal{C}(s_1, s_i, h-m-1) + \mathcal{C}(s_n, s_{k+1}, h-m-1)$.

Denoting by $B(\overleftarrow{s_i s_j}, m)$ the cost of range assignment with a right bridge $\overleftarrow{s_i s_j}$ where s_j is reached from s_α using m hops, we have $B(\overleftarrow{s_i s_j}, m) = Min(B_1, B_2)$.

Apart from identifying s_k, $B(\overleftarrow{s_i s_j}, m)$ can be calculated in O(1) time, because

(i) $\mathcal{C}(s_j, s_\alpha, m) = \mathcal{C}(s_\alpha, s_j, m) = M_3[m, j]$ (by Lemma 2),
(ii) $\mathcal{C}(s_1, s_i, h-m-1) = M_2[h-m-1, i]$,
(iii) $\mathcal{C}(s_n, s_k, h-m-1) = M_4[h-m-1, k]$, and
(iv) all these matrices are already available.

To get an optimal solution with a right-bridge, we need to find $Min_{i=1}^{\alpha-1}\ Min_{j=\alpha+1}^{n}\ Min_{m=1}^{h-1}\ B(\overleftarrow{s_i s_j}, m)$.

In our algorithm, we fix each s_i and compute $Min_{j=\alpha+1}^{n}\ Min_{m=1}^{h-1}\ B(\overleftarrow{s_i s_j}, m)$ using Lemma 7.

Lemma 7. *If* $s_j \in S \setminus \{s_1, s_2, \ldots, s_\alpha\}$, *then*

$$\mathcal{C}(s_j, s_\alpha, \mu-1) - \mathcal{C}(s_j, s_\alpha, \mu) \leqslant \mathcal{C}(s_{j+1}, s_\alpha, \mu-1) - \mathcal{C}(s_{j+1}, s_\alpha, \mu).$$

Proof. Let $A = \{a_0 = s_\alpha, a_1, a_2, \ldots, a_{\mu-2}\}$ denote the subsequence (radio stations) of S having non-zero ranges in $R(s_{j+1}, s_\alpha, \mu-1)$. We use $a_{\mu-1}$ to denote s_{j+1}. Thus, the range assigned to $a_i \in A$ is $(x(a_{i+1}) - x(a_i))$ for $i = 0, 1, 2, \ldots, \mu-2$. We use $cost(A)$ to denote $\mathcal{C}(s_{j+1}, s_\alpha, \mu-1)$. Again, let $B = \{b_0, b_1, b_2, \ldots, b_{\mu-1}\}$ denotes the set of radio-stations having non-zero ranges in $R(s_j, s_\alpha, \mu)$, i.e., $cost(B) = \mathcal{C}(s_j, s_\alpha, \mu)$. The ranges assigned to $b_i (\in B)$ is $(x(b_{i+1}) - x(b_i))$ for $i = 0, 1, 2, \ldots, \mu-1$.

Let us now observe the pairs (a_i, b_{i+1}), for $i = 0, 1, 2, \ldots, \mu-1$. Note that, $x(a_0) - x(b_1) < 0$, and $x(a_{\mu-1}) - x(b_\mu) > 0$. This implies, there exists at least one $i \in [1, 2, \ldots, \mu-1]$ such that $x(a_{i-1}) - x(b_i) \leqslant 0$ and $x(a_i) - x(b_{i+1}) \geqslant 0$. We consider the smallest $i \geqslant 1$ such that $x(a_i) - x(b_{i+1}) \geqslant 0$, and construct two subsequences of radio stations, namely $C = \{a_0 = b_0, a_1, \ldots, a_{i-1}, b_{i+1}, b_{i+2}, \ldots, b_{\mu-1}\}$ and $D = \{a_0 = b_0, b_1, b_2, \ldots, b_i, a_i, a_{i+1}, \ldots, a_{\mu-2}\}$, where length of C is $\mu-1$ and that of D is μ. The ranges assigned to the members in C and D are, respectively,

- $\{x(a_1) - x(a_0), \ldots, x(a_{i-1}) - x(a_{i-2}), x(b_{i+1}) - x(a_{i-1}), x(b_{i+2}) - x(b_{i+1}), \ldots, x(b_\mu) - x(b_{\mu-1})\}$, and
- $\{x(b_1) - x(b_0), \ldots, x(b_i) - x(b_{i-1}), x(a_i) - x(b_i), x(a_{i+1}) - x(a_i), \ldots, x(a_{\mu-1}) - x(a_{\mu-2})\}$.

The corresponding costs of these range assignments are

$$cost(C) = \sum_{j=0}^{j=i-2} (x(a_{j+1}) - x(a_j))^2 + (x(b_{i+1}) - x(a_{i-1}))^2 + \sum_{j=i+1}^{j=\mu-1} (x(b_{j+1}) - x(b_j))^2,$$

and

$$cost(D) = \sum_{j=0}^{j=i-1} (x(b_{j+1}) - x(b_j))^2 + (x(a_i) - x(b_i))^2 + \sum_{j=i}^{j=\mu-2} (x(a_{j+1}) - x(a_j))^2.$$

Thus,

$$cost(C) + cost(D) = \left(\sum_{j=0}^{j=\mu-2} (x(a_{j+1}) - x(a_j))^2 - (x(a_i) - x(a_{i-1}))^2 \right)$$

$$+\left(\sum_{j=0}^{j=\mu-1}(x(b_{j+1})-x(b_j))^2-(x(b_{i+1})-x(b_i))^2\right)$$
$$+(x(b_{i+1})-x(a_{i-1}))^2+(x(a_i)-x(b_i))^2$$
$$=cost(A)+cost(B)+2(x(a_i)-x(b_{i+1}))(x(a_{i+1})-x(b_i))\leqslant cost(A)+cost(B)$$

(due to the choice of i as mentioned above).

Again, $\mathcal{C}(s_j, s_\alpha, \mu-1)\leqslant cost(C)$ and $\mathcal{C}(s_{j+1}, s_\alpha, \mu)\leqslant cost(D)$. Thus, the lemma follows. □

Lemma 8. *While using the bridge $\overleftarrow{s_i s_j}$, $i<\alpha<j$, if $B(\overleftarrow{s_i s_j},\mu)\leqslant B(\overleftarrow{s_i s_j},\mu+1)$ then $B(\overleftarrow{s_i s_j},\mu+1)\leqslant B(\overleftarrow{s_i s_j},\mu+2)$.*

Proof. The gain in cost for increasing the number of hops from μ to $\mu+1$ to reach from s_α to s_j is $a_1=\mathcal{C}(s_j,s_\alpha,\mu)-\mathcal{C}(s_j,s_\alpha,\mu+1)\geqslant 0$. In order to maintain h-hop reachability from s_α to s_1 and s_n, we need to reach both from s_i to s_1 and from s_k to s_n using at most $h-\mu-2$ hops instead of $h-\mu-1$ hops. Thus, the amount of increase in the corresponding costs are $a_2=\mathcal{C}(s_1,s_i,h-\mu-2)-\mathcal{C}(s_1,s_i,h-\mu-1)\geqslant 0$ and $a_3=\mathcal{C}(s_n,s_k,h-\mu-2)-\mathcal{C}(s_n,s_k,h-\mu-1)\geqslant 0$.

As stated in the lemma, $B(\overleftarrow{s_i s_j},\mu)-B(\overleftarrow{s_i s_j},\mu+1)\leqslant 0$ implies $a_1-a_2-a_3\leqslant 0$.

Now, the gain in cost for increasing the number of hops from $\mu+1$ to $\mu+2$ to reach from s_α to s_j is $a_1'=\mathcal{C}(s_j,s_\alpha,\mu+1)-\mathcal{C}(s_j,s_\alpha,\mu+2)\geqslant 0$. This causes the reduction in number of hops from $h-\mu-2$ to $h-\mu-3$ for reaching s_1 from s_i and s_n from s_k. The loss in the corresponding costs are $a_2'=\mathcal{C}(s_1,s_i,h-\mu-3)-\mathcal{C}(s_1,s_i,h-\mu-2)\geqslant 0$ and $a_3'=\mathcal{C}(s_n,s_k,h-\mu-3)-\mathcal{C}(s_n,s_k,h-\mu-2)\geqslant 0$.

By Lemma 4, $a_1'\leqslant a_1$, $a_2'\geqslant a_2$ and $a_3'\geqslant a_3$. Thus,

$$B(\overleftarrow{s_i s_j},\mu+1)-B(\overleftarrow{s_i s_j},\mu+2)=a_1'-a_2'-a_3'\leqslant a_1-a_2-a_3\leqslant 0. \qquad \square$$

Lemma 8 implies that while using the right-bridge $\overleftarrow{s_i s_j}$, we vary the number of hops m to reach from s_α to s_j, and compute the corresponding cost $B(\overleftarrow{s_i s_j},m)$. As soon as $m=\mu$ is reached such that $B(\overleftarrow{s_i s_j},\mu)<B(\overleftarrow{s_i s_j},\mu+1)$, there is no need to check the costs by increasing m beyond $\mu+1$.

After computing the optimum range assignment with the right-bridge $\overleftarrow{s_i s_j}$, we proceed to compute the same with right-bridge $\overleftarrow{s_i s_{j+1}}$. The following lemma says that if the optimum $B(\overleftarrow{s_i s_j},m)$ is achieved for $m=\mu$ then while considering the right-bridge $\overleftarrow{s_i s_{j+1}}$, the optimum $B(\overleftarrow{s_i s_{j+1}},m)$ will be achieved for some $m\geqslant\mu$. Here, it needs to be mentioned that, we could not explore any relationship among the optimum costs of range assignments using $\overleftarrow{s_i s_j}$ and $\overleftarrow{s_i s_{j+1}}$.

Lemma 9. *For a given $s_i\in S$, $i<\alpha$, if $Min_{m=1}^h\ B(\overleftarrow{s_i s_j},m)$ and $Min_{m=1}^h\ B(\overleftarrow{s_i s_{j+1}},m)$ are achieved for $m=\mu$ and ν, respectively, then $\nu\geqslant\mu$.*

Proof. As s_i is fixed, we compute the optimal range assignment $R(s_1,s_i,h-m-1)$ to reach from s_i to s_1.

While using $\overleftarrow{s_i s_j}$, $\rho(s_j)=\delta(s_j,s_i)$, and this enables s_j to reach s_k to its right (i.e. $\delta(s_j,s_i)\geqslant\delta(s_j,s_k)$). Similarly, while using $\overleftarrow{s_i s_{j+1}}$, $\rho(s_{j+1})=\delta(s_{j+1},s_i)$, and this enables s_{j+1} to reach s_ℓ to its right (i.e. $\delta(s_{j+1},s_i)\geqslant\delta(s_{j+1},s_\ell)$). Here $j+1\leqslant k\leqslant\ell$.

In order to prove the lemma, we need only to show that $B(\overleftarrow{s_i s_{j+1}},\mu-1)\geqslant B(\overleftarrow{s_i s_{j+1}},\mu)$. By Lemma 8, this will automatically imply $B(\overleftarrow{s_i s_{j+1}},m-1)\geqslant B(\overleftarrow{s_i s_{j+1}},m)$ for all $m\leqslant\mu$. Thus, if $Min(B(\overleftarrow{s_i s_{j+1}},m))$ is achieved for $m=\nu$, then $\nu>\mu$.

To prove the above inequality, let $a_1=\mathcal{C}(s_j,s_\alpha,\mu-1)-\mathcal{C}(s_j,s_\alpha,\mu)$,

$$a_1'=\mathcal{C}(s_\alpha,s_{j-1},\mu-1)-\mathcal{C}(s_\alpha,s_{j-1},\mu),$$

$$a_2=\mathcal{C}(s_1,s_i,h-\mu-1)-\mathcal{C}(s_1,s_i,h-\mu),$$

$$a_3=\mathcal{C}(s_n,s_k,h-\mu-1)-\mathcal{C}(s_n,s_k,h-\mu)\text{ and,}$$

$$a_3'=\mathcal{C}(s_n,s_\ell,h-\mu-1)-\mathcal{C}(s_n,s_\ell,h-\mu).$$

As $B(\overleftarrow{s_i s_j}, \mu - 1) > B(\overleftarrow{s_i s_j}, \mu)$, we have $a_1 - a_2 - a_3 > 0$. By Lemma 7, $a'_1 \geqslant a_1$ and $a'_3 \leqslant a_3$. Hence, the amount of gain in cost for increasing the number of hops from $\mu - 1$ to μ for reaching from s_α to s_{j+1} and then using the bridge $\overleftarrow{s_i s_{j+1}}$ for the broadcast to the other nodes in S is equal to $B(\overleftarrow{s_i s_{j+1}}, \mu - 1) - B(\overleftarrow{s_i s_{j+1}}, \mu) = a'_1 - a_2 - a'_3 \geqslant a_1 - a_2 - a_3 \geqslant 0$. $\square$

Given a source-station s_α and another station s_i $i < \alpha$, the optimal range assignment of the members in S consisting of a functional right-bridge incident at s_i, can be computed using the following algorithm:

Algorithm Range_Assign_using_Right_Bridge(s_i)

Step 1: We initialize $OPT\_j = \alpha$, $OPT\_cost = \infty$ and $k\_store = \alpha$, and $\mu = 1$ (* μ stores the number of hops alloted to reach s_j from s_α *).
Start with $m = 1$ and $j = \alpha + 1$.
The role of *k_store* will be clear in the procedure **compute** invoked from this algorithm.
Step 2: At each j, we execute **compute**($B(\overleftarrow{s_i s_j}, m)$, *k_store*) by incrementing m from its current value upwards until
(i) $B(\overleftarrow{s_i s_j}, m) > B(\overleftarrow{s_i s_j}, m - 1)$ is achieved (see Lemma 8) or
(ii) m attains its maximum allowable value $Min(h - 2, j - \alpha)$.
Step 3: Update *OPT_cost* and *OPT_j* observing the value of $B(\overleftarrow{s_i s_j}, m - 1)$ or $B(\overleftarrow{s_i s_j}, m)$ depending on whether Step 2 has terminated depending on Case (i) or Case (ii).
Step 4: For the next choice of j, update μ by $m - 1$ or m depending on whether Case (i) or (ii) occurred in Step 2 (see Lemma 9).

Procedure **compute**($B(\overleftarrow{s_i s_j}, m)$, k_store)

- Initialize $k = k\_store$.
- Increment k to identify the right-most radio-station such that $\delta(s_j, s_k) \leqslant \rho(s_j) (= \delta(s_j, s_i))$.
- Set $k\_store = k$ for further use. (* i.e., for next j, the search for k will start from *k_store* *)
- Compute $B(\overleftarrow{s_i s_j}, m) = (\rho(s_j))^2 + R(s_j, s_\alpha, m) + R(s_1, s_i, h - m - 1) + R(s_n, s_k, h - m - 1)$; the last three terms are available in $M_3[m, j]$, $M_2[h - m - 1, i]$ and $M_4[h - m - 1, k]$ respectively.

Theorem 2. *For a given s_i ($i < \alpha$), algorithm Range_Assign_using_Right_Bridge needs* $O(n - \alpha + h)$ *time in the worst case.*

Proof. Follows from Lemmas 8 and 9, and the role of k_store in the procedure **compute** for locating rightmost s_k such that $\delta(s_j, s_k) \geqslant \rho(s_j)$. $\square$

4.4. Complexity analysis

Theorem 3. *Given a set of radio station S and a source station $s_\alpha \in S$, the optimum range assignment for broadcasting message from s_α to all the members in S using at most h-hops can be computed in* $O(n^2)$ *time and using* $O(nh)$ *space.*

Proof. Phase 1 needs $O(nh)$ time for initializing the matrices. Optimum functional bridge-free solution can be obtained in $O(n)$ time as described in Phase 2. Finally in Phase 3, we fix s_i to the left of s_α and identify the optimum solution with a functional right-bridge incident at s_i in $O(n - \alpha + h)$ time (see Theorem 2). For $(\alpha - 1)$ such s_i's, the total time required in this phase is $O(\alpha \times (n - \alpha + h))$. Similarly, the worst case time required for finding the optimum range assignment with exactly one functional left-bridge is $O((n - \alpha) \times (\alpha + h))$. Thus, the result follows. $\square$

Note. The interesting question is whether one can design an efficient algorithm for Phase 3 such that the time complexity can further be reduce to $O(nh \times polylog(h))$.

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